Solⁿ - 24/10/2023

① $e^y + xy = e$ — ①

Put $x = 0$

$e^y = e^1$

$y = 1$

$\boxed{\text{Point } (0,1)}$

$$e^y \frac{dy}{dx} + x\frac{dy}{dx} + y = 0 \quad \text{— ②}$$

Putting $x=0,\ y=1$

$$e\frac{dy}{dx} + 0 + 1 = 0$$

$$\boxed{\frac{dy}{dx} = \frac{-1}{e}}$$

$$e^y \frac{d^2y}{dx^2} + \frac{dy}{dx}\cdot e^y \frac{dy}{dx} + x\cdot\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{dy}{dx} = 0$$

$$e^y \frac{d^2y}{dx^2} + e^y\left(\frac{dy}{dx}\right)^2 + x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0 \quad \text{— ③}$$

Put $x=0,\ y=1,\ \frac{dy}{dx} = \frac{-1}{e}$

$$e\frac{d^2y}{dx^2} + e\cdot\frac{1}{e^2} + 0 + 2\cdot\frac{-1}{e} = 0$$

$$e\frac{d^2y}{dx^2} + \frac{1}{e} + 0 + \frac{-2}{e} = 0$$

$$e\frac{d^2y}{dx^2} = \frac{1}{e}$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{1}{e^2}}$$

$$\left(\frac{dy}{dx}, \frac{d^2y}{dx^2}\right) \equiv \left(-\frac{1}{e}, \frac{1}{e^2}\right)$$

option b

② $\tan^{-1}\left(\frac{\sin x - \cos x}{\sin x + \cos x}\right)$

$$= \tan^{-1}\left(\frac{\tan x - 1}{\tan x + 1}\right)$$

$$= -\tan^{-1}\left(\tan\left(\frac{\pi}{4} - x\right)\right)$$

$$f(x) = -\left(\pi/4 - x\right)$$

$$= x - \pi/4$$

$$\boxed{f'(x) = 1}$$

$$y = \frac{x}{2}$$

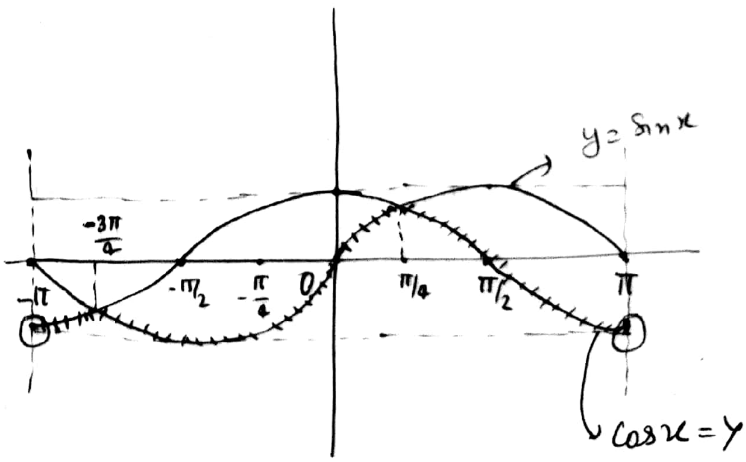
$$\frac{dy}{dx} = \frac{1}{2}$$

$\tan^{-1}\left(\frac{\sin x - \cos x}{\sin x + \cos x}\right)$ w.r.t. $x/2$

$$= \frac{1}{1/2} = 2$$

option d

③ $f(x) = \min\{\sin x, \cos x\}$



$f(x)$ is not differentiable at $x = \frac{-3\pi}{4}, \frac{\pi}{4}$

$$\Rightarrow S \subseteq \left\{-\frac{3\pi}{4}, \frac{-\pi}{4}, \frac{3\pi}{4}, \frac{\pi}{4}\right\}$$

option b

④ $(2x)^{2y} = 4e^{2x-2y}$

Taking log on both sides

$2y \log 2x = \log 4 + 2x - 2y$ — ①

$2y \cdot \frac{1}{2x} \cdot 2 + 2\log 2x \frac{dy}{dx} = 2 - 2\frac{dy}{dx}$

$2\frac{y}{x} + 2\log 2x \cdot \frac{dy}{dx} = \left(1 - \frac{dy}{dx}\right)2$

$2 \cdot \log 2x \frac{dy}{dx} + 2 \cdot \frac{dy}{dx} = 2 - \frac{2y}{x}$

$2(\log 2x + 1)\frac{dy}{dx} = 2 - \frac{2y}{x}$

$2 \cdot \frac{dy}{dx}(\log 2x + 1) = \frac{2x - 2y}{x}$ — ⑪

from ①

$\frac{2y}{x}\log 2x = \frac{\log 4}{x} + \frac{2x - 2y}{x}$ — ⑪⑪

$\frac{2y}{x}\log 2x = \frac{2\log 2}{x} + 2\frac{dy}{dx}(\log 2x + 1)$

$\frac{y}{x}\log 2x = \frac{\log 2}{x} + \frac{dy}{dx}(\log 2x + 1)$ — ⑪V

$2y(\log 2x + 1) = 2\log 2 + 2x$

$y(\log 2x + 1) = \log 2 + x$

$y = \frac{\log 2 + x}{(\log 2x + 1)}$

$\frac{(\log 2 + x)\log 2x}{x(\log 2x + 1)} = \frac{\log 2}{x} + \frac{dy}{dx}(\log 2x + 1)$

$\frac{(\log 2 + x)\log 2x}{x \log 2x + 1)} - \frac{\log 2}{x} = \frac{dy}{dx}(\log 2x + 1)$

$\frac{\log 2 \cdot \log 2x + x \cdot \log 2x - \log 2 \cdot (\log 2x + 1)}{x(\log 2x + 1)} = \frac{dy}{dx}(\log 2x + 1)$

$\frac{\cancel{\log 2 \cdot \log 2x} + x \cdot \log 2x - \cancel{\log 2 \cdot \log 2x} - \log 2}{x} =$

$(\log 2x + 1)^2 \frac{dy}{dx}$

$(\log 2x + 1)^2 \frac{dy}{dx} = \frac{x \log 2x - \log 2}{x}$

option a

⑤ $f(x) = x^3 + ax^2 + bx + c$

$f'(x) = 3x^2 + 2ax + b$

$f'(1) = 3 + 2a + b$

$f''(x) = 6x + 2a$

$f''(2) = 12 + 2a$

$f'''(x) = 6 \Rightarrow f'''(3) = 6$

$f(x) = x^3 + x^2 f'(1) + x f''(2) + f'''(3)$

$f'(1) = a \Rightarrow 3 + 2a + b = a$

$\Rightarrow a + b = -3$

$f''(2) = b \Rightarrow 12 + 2a = b$

$\Rightarrow 2a - b = -12$

$f'''(3) = c = 6$

$a + \cancel{b} = -3$

$2a - \cancel{b} = -12$

$3a = -15 \Rightarrow a = -5$

$b = 2$

$f(x) = x^3 - 5x^2 + 2x + 6$

$f(2) = 8 - 20 + 4 + 6$

$= 18 - 20$

$= -2$

option c

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⑥ $\binom{30}{0}\binom{30}{10} - \binom{30}{1}\binom{30}{11} + \binom{30}{2}\binom{30}{12} + \cdots \binom{30}{20}\binom{30}{30}$

where $\binom{n}{r} = {}^nC_r$

$^{30}C_0\,^{30}C_{10} - {}^{30}C_1\,^{30}C_{11} + {}^{30}C_2\,^{30}C_{12} - \cdots {}^{30}C_{20}\,^{30}C_{30}$

$(1+x)^{30} = {}^{30}C_0 + {}^{30}C_1 x + {}^{30}C_2 x^2 + \cdots {}^{30}C_{20}x^{20} + \cdots {}^{30}C_{30}x^{30}$ — ①

$(x-1)^{30} = {}^{30}C_0 x^{30} - {}^{30}C_1 x^{29} + \cdots + {}^{30}C_{10}x^{20} - {}^{30}C_{11}x^{19} + {}^{30}C_{12}x^{18} + \cdots {}^{30}C_{30}x^0$ — ②

Multiplying equation ① and ②, we get

$(x^2-1)^{30} = \left({}^{30}C_0 + {}^{30}C_1 x + {}^{30}C_2 x^2 + \cdots + {}^{30}C_{20}x^{20} + \cdots + {}^{30}C_{30}x^{30}\right)$
$\cdot\left({}^{30}C_0 x^{30} - {}^{30}C_1 x^{29} + \cdots + {}^{30}C_{10}x^{20} - {}^{30}C_{11}x^{19} \cdots {}^{30}C_{30}x^0\right)$

Equating the coefficient of x^{20} on both sides

$^{30}C_{10} = {}^{30}C_0 \cdot {}^{30}C_{10} - {}^{30}C_1\,^{30}C_{11} + {}^{30}C_2\,^{30}C_{12} - \cdots {}^{30}C_{20}\,^{30}C_{30}$

option a

⑦ $2\cdot{}^{20}C_0 + 5\cdot{}^{20}C_1 + 8\cdot{}^{20}C_2 + 11\cdot{}^{20}C_3 + \cdots + 62\,^{20}C_{20}$

$= \sum_{r=0}^{20} (3r+2)\,^{20}C_r$

$= 3\sum_{r=0}^{20} r\cdot{}^{20}C_r + 2\sum_{r=0}^{20} {}^{20}C_r$

$= 60\sum_{r=1}^{20} {}^{19}C_{n-1} + 2\sum_{r=0}^{20} {}^{20}C_r$

$= 60 \times 2^{19} + 2\times 2^{20}$

$= 2\times 2\times 15\times 2^{19} + 2^{21}$

$= 2^{21}(15+1) = 2^{21}\times 16 = 2^{25}$

option b

⑧ $\left(\frac{2}{x} + x^{\log_8 x}\right)^6$

given $T_4 = 20\times 8^7$

$\Rightarrow {}^6C_3 \left(\frac{2}{x}\right)^3 \times \left(x^{\log_8 x}\right)^3 = 20\times 8^7$

$$8 \times \cancel{20} \times \left(\frac{x^{\log_8 x}}{x}\right)^3 = \cancel{20} \times 8^7$$

$$\left(\frac{x^{\log_8 x}}{x}\right)^3 = 8^6$$

$$\frac{x^{\log_8 x}}{x} = 64 = 8^2$$

Taking $\log_8$ on both sides then

$$\log_8 x^{\log_8 x} - \log_8 x = 2 \log_8 8$$

$$\log_8 x \cdot \log_8 x - \log_8 x = 2$$

$$\log_8 x \left(\log_8 x - 1\right) = 2$$

$$(\log_8 x)^2 - \log_8 x - 2 = 0$$

Put $\log_8 x = t$

$$t^2 - t - 2 = 0$$

$$t^2 - 2t + t - 2 = 0$$

$$t(t-2) + 1(t-2) = 0$$

$$(t+1)(t-2) = 0$$

$$t = -1, \quad t = 2$$

$$\log_8 x = -1, \quad \log_8 x = 2$$

$$x = \frac{1}{8}, \qquad x = 8^2$$

Option b

(9) $(1 + x^{\log_2 x})^5$

$$T_3 = 2560$$

$$5C_2 \left(x^{\log_2 x}\right)^2 = 2560$$

$$5C_2 \left(x^{\log_2 x}\right)^2 = 2560$$

$$\left(x^{\log_2 x}\right)^2 = 256 = (16)^2$$

$$x^{\log_2 x} = 16 = 2^4$$

Taking $\log_2$ on both side

$$\log_2 x^{\log_2 x} = \log_2 2^4$$

$$(\log_2 x)^2 = 4$$

$$\log_2 x = \pm 2$$

$$x = 2^2 = 4 \quad \text{or} \quad x = 2^{-2} = \frac{1}{4}$$

Option a

(10) $\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$

$$T_{r+1} = {}^9C_r \left(\frac{3}{2}x^2\right)^{9-r} \left(-\frac{1}{3x}\right)^r$$

$$= {}^9C_r \left(\frac{3}{2}\right)^{9-r} x^{18-2r} (-1)^r (3)^{-r} x^{-r}$$

For the term independent of x

$$18 - 3r = 0$$

$$3r = 18 \Rightarrow r = 6$$

$$T_7 = {}^9C_6 \left(\frac{3}{2}\right)^{9-6} \left(-\frac{1}{3}\right)^6$$

$$= {}^9C_6 \left(\frac{3}{2}\right)^3 \left(\frac{1}{3}\right)^6$$

$$= \frac{3\cancel{9} \times \cancel{8}^{4} \times 7}{\cancel{6}\,2} \cdot \frac{1}{2^3 \times 3^3}$$

$$= \frac{\cancel{3} \times \cancel{4}^{2} \times 7}{\cancel{2} \times \cancel{3} \times 3^2 \cdot 2^2} = \frac{7}{18}$$

$$18k = 18 \times \frac{7}{18} = 7$$

Option C